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Non-perturbative RR potentials in the $\hat{c} = 1$ matrix model

David J. Gross and Johannes Walcher

Kavli Institute for Theoretical Physics, University of California
Santa Barbara, CA 93106, U.S.A.
E-mail: `gross@itp.ucsb.edu`, `walcher@kitp.ucsb.edu`

Abstract: We use the $\hat{c} = 1$ matrix model to compute the potential energy $V(C)$ for (the zero mode of) the RR scalar in two-dimensional type 0B string theory. The potential is induced by turning on a background RR flux, which in the matrix model corresponds to unequal Fermi levels for the two types of fermions. Perturbatively, this leads to a linear runaway potential, but non-perturbative effects stabilize the potential, and we find the exact expression $V(C) = \frac{1}{2\pi}\int da \, \arccos\bigl[\cos(C)/\sqrt{1+e^{-2\pi a}}\bigr]$. We also compute the finite-temperature partition function of the 0B theory in the presence of flux. The perturbative expansion is T-dual to the analogous result in type 0A theory, but non-perturbative effects (which depend on C) do not respect naive $R \to 1/R$ duality. The model can also be used to study scattering amplitudes in background RR fluxes.

Keywords: String Duality, Matrix Models.

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Contents

1. Introduction

The stabilization of moduli is an important problem on the way towards constructing realistic models in string/M-theory. It has been appreciated in recent years that a simple way of stabilizing moduli is to turn on fluxes in the compactification manifold. For example, in constructions based on the type IIB string or F-theory, turning on fluxes induces a superpotential which generically fixes all complex structure moduli [1]. However, the Kähler moduli and their associated RR partners remain untouched in this procedure, and stabilizing them seems to require non-perturbative effects [2]. More precisely, all potentials that can be induced at the perturbative level (by breaking supersymmetry) are runaway potentials, without any stable stationary point. But after including non-perturbative effects [3], the potential can have (meta-)stable minima, of great cosmological interest [2].

In this paper, we study the potential for a particularly simple "modulus", the zero mode of the Ramond-Ramond (RR) scalar C of type 0B "non-critical" string theory in two dimensions. This constant part of C is related to a shift symmetry that is analogous to the gauge symmetries of p-form fields in higher-dimensional string theories. But while in higher dimensions, potentials for the periods of these p-form gauge fields are sometimes hard to compute due to the lack of a fully non-perturbative definition of string/M-theory, the advantage of the two-dimensional setup is that there exists an accessible non-perturbative definition of the theory, the so-called $\hat{c} = 1$ matrix model [5, 6]. We will therefore be able to compute everything exactly.

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As we will see, the mechanism for generating the potential in the two-dimensional 0B context is essentially similar to the situation in higher dimensions. One can induce a potential for C by turning on a RR background flux. Perturbatively, this potential would exhibit runaway, but this behavior is stabilized by non-perturbative effects. The exact expression is (see figure 4 on page 13 for a graph of this function)

$$V(C) = \frac{1}{2\pi} \int_{\mu-Q}^{\mu+Q} da \arccos\left[\frac{\cos C}{\sqrt{1+e^{-2\pi a}}}\right], \tag{1.1}$$

where μ is the tachyon background (worldsheet cosmological constant) determining the string coupling, and Q is the appropriately normalized RR flux. The crucial step in understanding $V(C)$ is the identification, in the finite-N matrix model, of the RR flux and the zero mode of C. Briefly, while μ corresponds to the distance of the Fermi level for the fermionic eigenvalues from the top of the inverted harmonic oscillator potential of the double-scaled matrix model, Q corresponds to the difference of the Fermi levels for the even and odd modes of the inverted harmonic oscillator.[1] Moreover, C is identified with the $\mathbb{Z}_2$-breaking boundary condition in the asymptotic regions of the potential. We will give a careful derivation of these identifications in the upcoming section 2.

In section 3, we then turn to the computation of the potential energy $V(C)$ as a function of the string coupling and background flux. Since the finite piece (1.1) vanishes at $Q = 0$, we will for completeness also compute the subleading contribution that vanishes when we take the double-scaling limit.

We then give two other applications of our identifications of section 2. In section 4, we compute the finite-temperature partition function for non-zero background flux Q, and compare with the analogous results from the type 0A matrix model. We find that once the 0A flux is continued to imaginary values, the perturbative expansions of the two results are T-dual to each other. The non-perturbative parts of the partition functions, however, are not mapped to each other under naive $R \to 1/R$ duality. We speculate on the meaning of this result. On the other hand, our results are invariant under "S-duality", which involves $\mu \to -\mu$ and dualization of $C \to \tilde{C}$. In addition, we find a novel duality which exchanges NS and RR background $\mu \leftrightarrow Q$. In the matrix model, these dualities are simple particle/hole dualities for the even and odd modes of the potential. Some of the relevant computations, such as the dependence of the partition function on C, and the specialization of the results to the "self-dual" radius $R = 1$, can be found in the appendix. In section 5, we study the bosonization of the matrix model eigenvalues when the Fermi levels for even and odd modes are not the same, and compute some S-matrix elements, following [7, 8]. We conclude in section 6 with a list of open problems.

For reviews of matrix models useful for our present discussion, see [9, 10, 11, 12]. More recent matrix model literature includes [13]–[30].

[1] This is the non-perturbative definition. Perturbatively, it can be reduced to the difference of the Fermi levels on the two sides of the potential, or for left and right moving eigenvalues, depending on the sign of μ. This identification was proposed in [6].

2. The 0B matrix model and its deformations

The matrix model which in the double-scaling limit describes two-dimensional $\mathcal{N} = 1$ supergravity coupled to $\hat{c} = 1$ matter with type 0B GSO projection [5, 6] is essentially similar to the conventional or "old-fashioned" matrix model describing two-dimensional bosonic gravity coupled to $c = 1$ matter. The definition begins with the quantum mechanics of an $N \times N$ hermitian matrix M with potential $V(M)$ which has a quadratic maximum at $M = 0$. In the singlet sector, this quantum mechanics can be reduced to the dynamics the matrix' eigenvalues λ behaving as N free fermions moving in the potential $V(\lambda)$. The continuum limit involves taking N to infinity, sending Planck's constant to zero, and adjusting the various parameters such that the Fermi level approaches the top of the potential to order $\hbar$. The only parameters surviving the limit are the Fermi level μ and the curvature of the potential at the maximum. The bosonic and the supersymmetric model differ slightly but importantly in both.

In the "old-fashioned" way of thinking [31, 32, 33], the expansion of the path-integral for finite N captures the discretization of the two-dimensional string worldsheet. The double-scaling limit is taken in order to tune the model to criticality. In the "reloaded" form [4], one thinks of the quantum mechanics as arising holographically as the world-volume theory on a stack of a large number of unstable D0-branes. In this context, the curvature of the potential at the maximum is identified with the mass of the open string tachyon. It is $m^2 = -1/\alpha'$ in the bosonic case and $m^2 = -1/2\alpha'$ in the supersymmetric case.

Also, in the old days of the bosonic model, fermions were only filled on one side of the maximum, which can only make sense in the perturbative $\hbar$ expansion. For the supersymmetric version, we are now instructed to fill both sides of the potential, and to interpret this non-perturbatively stable model as the two-dimensional linear dilaton background of the type 0B string. As in the bosonic case, collective excitations of the fermionic eigenvalues are identified with perturbative string theory degrees of freedom. In the bosonic case, there is only the center of mass of the string, which in two dimensions is a massless tachyon T. In the supersymmetric 0B model, one has in addition a massless RR scalar C. According to [6, 5], even fluctuations of the Fermi sea correspond to T, and odd fluctuations correspond to C. In other words, the $\mathbb{Z}_2$ parity symmetry of the matrix model $M \to -M$ is identified with $(-1)^{F_L}$ in the string theory picture.

In this paper, we will be interested in studying aspects of the RR sector of this 0B matrix model. As we have mentioned in the introduction, the RR scalar shares many interesting features with the RR p-form fields familiar from higher-dimensional string theories. For type 0 theories in general, the spacetime effective action contains to lowest order terms of the form [34, 35, 36, 37, 6]

$$\frac{1}{8\pi} \int f(T)\, dC \wedge *dC \tag{2.1}$$

that couple the RR forms to the closed string tachyon T. Constraints on the functions $f(T)$ can be obtained from the requirement of T-duality [37], and suggest taking $f(T) = e^{-2T}$, as we will do, following [6]. In two dimensions, there is only one RR form, which is a (nonchiral) scalar boson. As usual, the zero mode of C does not appear in the action (2.1), and we have a one-parameter shift symmetry. Similarly to higher dimensions, this perturbative gauge symmetry is violated non-perturbatively by instanton effects. Such instantons should generically generate a potential for C, and by analogy with the Yang-Mills θ-parameter, one might expect that this potential has to lowest order the form $e^{-1/g_s}\cos C$. We will see that the details are in fact somewhat different in the present case.

The natural field strength associated with C is $F = e^{-T} dC$, satisfying the Bianchi identity and equation of motion

$$\begin{aligned} d\big(e^{T} F\big) &= 0\,, \\ d\big(e^{-T} *F\big) &= 0\,. \end{aligned} \tag{2.2}$$

Recall that to lowest order, the tachyon background is given by

$$T(\phi,t) = T(\phi) = \mu e^{\phi}\,, \tag{2.3}$$

where ϕ is the spacelike Liouville direction supporting the linear dilaton. In this background the linearized equations of motion (2.2) for F admit the two linearly independent solutions [6],

$$F = e^{-T} dt\,, \tag{2.4}$$

$$F = e^{T} d\phi\,, \tag{2.5}$$

to which we shall refer to as the electric and magnetic solution, respectively. Of course, since C is the middle-dimensional form in the two-dimensional 0B theory, we could equally well have chosen to describe this using the dual form $\tilde{C}$ related to C by $e^{-T} dC = F = *\tilde{F} = e^{T} * d\tilde{C}$, so that the action for $\tilde{C}$ is (see [37] for some details on electric-magnetic dualities in type 0 theories)

$$\frac{1}{8\pi}\int e^{2T}\, d\tilde{C}\wedge *d\tilde{C}\,. \tag{2.6}$$

We also note that depending on the sign of μ, one of the two solutions (2.4), (2.5) decays at infinity $\phi\to\infty$, which as we recall is the asymptotic region where the matrix model lives. For clarity, we will primarily discuss the case $\mu > 0$, by which we mean that the Fermi level is below the top of the potential. In this case, only the electric solution (2.4) is physically relevant.[2] But as is apparent from our discussion, one in facts expects that the theory enjoys an exact S-like duality $\mu\to-\mu$, $C\to\tilde{C}$. We will confirm that after a proper identification of C and $\tilde{C}$, all our results are invariant under this duality.

In [6], it was proposed to identify the generator of the shift symmetry of C with the perturbatively (for $\mu > 0$) well-defined difference in the number of fermions on the right and on the left of the maximum of the potential in the matrix model. One can justify this identification from the fact that instantons in spacetime are related to the tunneling of

[2]Our conventions are slightly different from the ones used in [6], where the magnetic solution is physical when the Fermi level is below the top of the potential. In other words, the field C of [6] is our magnetic variable $\tilde{C}$.

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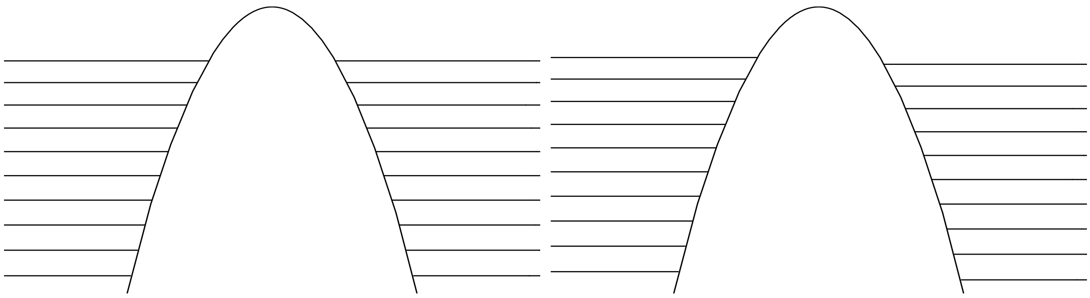

Figure 1: Perturbative energy levels in the inverted harmonic oscillator potential (finite N). For symmetric boundary conditions ($C = 0$), the spectrum is doubly degenerate (left panel). Changing C shifts the phase of fermions on the left and right of the potential. This raises the levels on the left and lowers them on the right (the right panel is for $C = 1$, where the periodicity of C is 2π). The energy shifts cancel out in the sum and C is perturbatively an exact symmetry in the double-scaling limit.

eigenvalues through the potential barrier in the matrix model. The instantons carry C-field charge and the tunneling changes the number of fermions on the left and right by one unit. Thus, a constant shift $C \to C + 2\alpha$ shifts the phase of the wavefunction of fermions on the left of the potential by $-\alpha$ and the phase of the wavefunction of fermions on the right by $+\alpha$.

Another identification provided in [6] is that the decaying solution (2.4) is related to the difference of the Fermi level for fermions on the left and the right of the potential, which perturbatively is again a well-defined notion. But since we are now allowing eigenvalue tunneling, one is naturally led to wonder about the non-perturbative stability of this flux background. From the outset, we notice that the total flux associated with the solution (2.5), $\int_{-\infty}^{\infty} F_t \, d\phi$, as well as the energy, $\int F_t^2 \, d\phi$, diverge as the volume of space. One the other hand, an instanton carries a quantized (electric) charge, and a single tunneling can only change the flux by a finite amount. Moreover, in the spacetime picture the instanton corresponds to a D-brane which is localized in the Liouville direction at infinite ϕ, and has no zero mode. It is therefore not clear whether instantons are actually able to destroy the flux completely. We will show in this paper that while the instantons do affect the identification of the RR sector in the matrix model, the flux background (2.4) is in fact perfectly stable.

We begin in the perturbative picture and consider the energy levels of the matrix eigenvalues close to the top of the potential, see figure 1. These energy levels are determined semiclassically by a systematic WKB expansion. To lowest order, we have the Bohr-Sommerfeld quantization condition for the n-th energy level ϵ_n,

$$\pi n \hbar = \int^{\lambda_*} \sqrt{2(\epsilon_n - V(\lambda))} d\lambda \,, \tag{2.7}$$

where we integrate up to the classical turning point λ_* satisfying $V(\lambda_*) = \epsilon_n$. The boundary condition at infinity depends on the details of the potential in this non-universal region, but can at most contribute an overall phase to the WKB approximation. If the boundary condition is $\mathbb{Z}_2$ symmetric, the energy levels on the left and right will match, $\epsilon_{L,n} = \epsilon_{R,n}$,

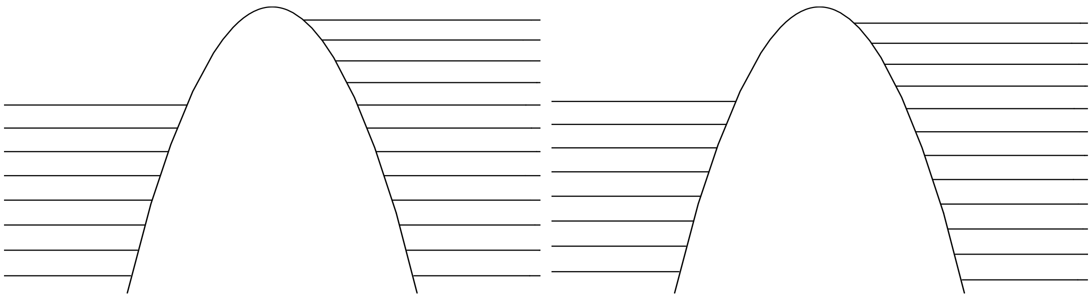

Figure 2: Perturbatively, we can fill the energy levels asymmetrically on the left and right of the potential, but keep the difference of Fermi levels finite in the double-scaling limit. This induces a linear potential for C.

as shown on the left in figure 1. We also record the familiar expression for the asymptotic density of states ρ, which is obtained from (2.7) upon approximating $V(\lambda)$ quadratically around 0 with $V''(0) = -1$,

$$\rho = \left|\frac{dn}{d\epsilon}\right| = \frac{\ln 2\pi\hbar n}{2\pi\hbar}\,. \tag{2.8}$$

Shifting the phase of the wavefunction by α corresponds to adding $\hbar\alpha$ on the r.h.s. of (2.7). In the convenient normalization $\alpha \sim C/2$, a shift of C by 2π maps the n-th energy level on the right to the $(n-1)$st and the n-th energy level on the left to the $(n+1)$st. More precisely, we find from

$$\frac{dn}{dC} = \frac{1}{2\pi} \tag{2.9}$$

that shifting C changes $\epsilon_{L/R}$ by

$$\frac{d\epsilon_{L/R}}{dC} = \pm\frac{1}{2\pi\rho}\,. \tag{2.10}$$

The two contributions cancel in the sum, so that if the energy levels are filled to the same Fermi level on the left and on the right, shifting C does not cost any energy, even before taking the double-scaling limit. In the double-scaling limit, which involves ρ diverging as $|\ln\hbar|$, a finite difference of fermion number or a finite shift of C does not change the renormalized Fermi level μ, which is measured from the top of the potential in units of $\hbar$. Moreover, since we neglect tunneling, the scattering amplitudes do not feel any mismatch of energy levels on the left and right of the potential. We conclude that C is perturbatively an exact symmetry in the double-scaling limit.

These considerations suggest a simple way of inducing a non-vanishing potential for C. The idea is to fill the eigenvalues asymmetrically, on the left up to μ_L, and on the right up to μ_R, and to keep the difference $\mu_L - \mu_R$ finite in the double-scaling limit, see figure 2. We will parameterize $\mu_L = \mu + Q$ and $\mu_R = \mu - Q$. In the double-scaling limit, non-zero Q makes two contributions to the energy. There is first of all an infinite contribution that arises because we have lifted an infinite number (in the limit) of fermions by a finite amount. This contribution will diverge as usual as $|\ln\hbar|$, which is identified with the volume of space in the spacetime picture. But there is also a finite contribution, which depends on C. Indeed, upon integrating (2.10) we find the contribution

$$V_{\rm pert}(C) = \int_{\mu_R}^{\mu_L} d\epsilon\, \rho\, \big(\epsilon_R(C) - \epsilon_R(0)\big) = -\frac{CQ}{\pi}\,. \tag{2.11}$$

(The derivation is similar for the other sign of Q.) We note that this term is non-zero but finite (not infinite) in the double-scaling limit.

One may wonder what kind of term in the spacetime action could capture a contribution of the form (2.11). Following the suggestion made in [6], we assume that Q is related to the flux of the RR one-form field strength, which in our conventions is the electric solution (2.4), $F = e^{-T}dC \propto Qe^{-T}dt$. We can then write (2.11) as

$$V_{\rm pert}(C) \propto \big(CF_t(+\infty) - CF_t(-\infty)\big) = \int d\phi\, \partial_\phi(CF_t)\,, \tag{2.12}$$

which corresponds to a total derivative term in the action of the form

$$\int d(CF)\,. \tag{2.13}$$

This term (2.13) is reminiscent of Chern-Simons terms familiar from higher-dimensional string theories. It would be interesting to see whether one can check the presence of such a term from the spacetime or Liouville point of view. The potential (1.1) we will compute can be viewed as the non-perturbative completion of the expression (2.13).

As we have mentioned, the potential energy (2.11) is linear in C and does not have any stationary point. In addition, the flux background in figure 2 appears to be unstable to eigenvalue tunneling. We will now give a qualitative description of how both C and the flux are stabilized once we include non-perturbative effects, and will present the quantitative calculations in the next section.

The picture is quite simple (see figure 3). If we allow for tunneling, the energy levels on the left and right of the potential mix, and the true eigenstates of the inverted harmonic oscillator are linear combinations of the two. For symmetric boundary conditions ($C = 0$), the eigenstates split into even and odd under $\lambda \to -\lambda$, with splitting (at finite N) of order $e^{-\pi a}$, where a is the distance from the top of the potential in units of $\hbar$. Each eigenfunction is equally supported on the left and right of the potential. When C is non-zero, the perturbative energy levels on the left and right do not match (see figure 1) and mixing is suppressed. Therefore, the eigenfunctions on each side of the potential essentially keep their identity, with small admixture from the other side. The eigenvalues depend linearly on C, as in eq. (2.10). When C reaches π, the perturbative energy levels would cross, leading to eigenvalue repulsion. The eigenfunctions are again equally supported on both sides of the potential. From $C = \pi$ to $C = 2\pi$, the picture is the same with inverted sign, and the full periodicity is $C \to C + 2\pi$. This dependence of the eigenvalues on C is depicted in figure 3.

Even though for $C \neq 0$, the eigenstates of the inverted harmonic oscillator are strictly speaking neither even nor odd, we will simply refer to the states by the parity they have once adiabatically continued to $C = 0$. An equivalent notion is the parity of the wavefunction

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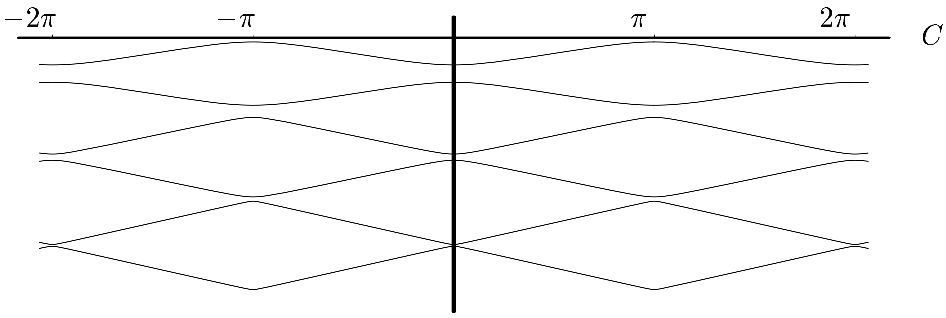

Figure 3: Energy eigenvalues of the inverted harmonic oscillator potential (at finite N) as a function of the boundary condition asymmetry C.

under the combined reflection $\lambda \to -\lambda$, $C \to -C$. But one has to keep in mind that except for small regions of order $e^{-\pi a}$ around $C = 0$ and π, these wavefunctions are actually well supported on one side of the potential.

Putting everything together, we are now led to the identification of the electric RR flux background Q with the difference of the Fermi levels for the even and odd eigenmodes of the inverted harmonic oscillator. Perturbatively, and for C outside the small regions around 0 and π, this definition reduces to the earlier one given in [6]. To be specific, we will put the Fermi level for the even modes at $\mu_+ = \mu + Q$ and for the odd modes at $\mu_- = \mu - Q$, and we assume for the time being that both μ_+ and μ_- are positive. The RR background thus corresponds to a natural deformation of the 0B matrix model, and is perfectly stable. Moreover, once Q is non-zero, we induce a potential energy $V(C)$ for the zero mode of C, which is obtained by summing up the dependence of the unpaired even/odd eigenvalues from μ_- to μ_+. Before we turn to the computation of $V(C)$, we would like to tie up a few loose ends of this identification of the RR flux in the matrix model.

A simple heuristic way of justifying the perturbative identification of the flux as the difference between Fermi levels on the left and on the right of the potential is as follows. According to [4, 5, 6], we can think of the matrix model holographically as the worldvolume theory on a stack of N unstable D0-branes of type 0B string theory. In this context, the matrix M can be identified with the open string tachyon $\mathcal{T}$ of the D0-branes, and the inverted harmonic oscillator is the maximum of the tachyon potential $\mathcal{V}(\mathcal{T})$. As is by now quite well established, open string tachyons on unstable D-branes also couple to RR fields via terms of the form [42]

$$\int \mathrm{Tr}\, d\mathcal{F}(\mathcal{T}) \wedge C\,. \tag{2.14}$$

The purpose of this coupling is to give the correct RR charge to a tachyonic kink interpolating between the two minima of $\mathcal{V}$, with the BPS condition $\Delta\mathcal{F} = \int \mathcal{V}$ relating charge and mass. The couplings $\mathcal{V}$ and $\mathcal{F}$ are not known in general — in particular, they might depend on the closed string tachyon in type 0B theory. However, all we need to retain presently is the behavior near the maximum of $\mathcal{V}$. Then, since the constant part of $\mathcal{F}$ does not matter, all we need to know is that while $\mathcal{V}$ is generically an even function under $\mathcal{T} \to -\mathcal{T}$, $\mathcal{F}$ is generically an odd function.

Eq. (2.14) is also relevant for describing the behavior of unstable D-branes in an external field. Upon integration by parts, we obtain [4]

$$\int \mathrm{Tr}\, \mathcal{F}(\mathcal{T}) \wedge \hat{F} \tag{2.15}$$

where $\hat{F} = dC$ is the field strength associated with C. Imagine we start (at finite N, so $\mathcal{V}$ is bounded from below) from a symmetric situation with equal Fermi levels on the left and on the right, and then turn on the RR flux. To first order, $\mathcal{F}$ is simply a linear function, so this shifts the maximum of the potential. If we neglect tunnneling, this corresponds to a difference between Fermi levels on the left and right of the maximum. The effects of tunneling, as we have explained above, depend on the zero mode of C, which is a boundary condition not included in (2.15).

It would be very nice to make this identification more rigorous, based either on a discretized worldsheet with non-vanishing RR flux or on decay amplitudes of D0-branes in such a background. In any case, we can summarize the situation by writing the second quantized hamiltonian

$$H = \int_{\lambda_L}^{\lambda_R} d\lambda \, \left[\frac{1}{2} \frac{\partial \Psi^\dagger(\lambda)}{\partial \lambda} \frac{\partial \Psi(\lambda)}{\partial \lambda} - \frac{\lambda^2}{2} \Psi^\dagger(\lambda)\Psi(\lambda) + \mu \Psi^\dagger(\lambda)\Psi(\lambda) + Q\Psi^\dagger(\lambda)\Psi(-\lambda) \right] , \tag{2.16}$$

where for definiteness, we have introduced boundary conditions by cutting off the integral at $\lambda_{L,R} = \mp 1 + \hbar C/2$. The equations of motion that follow from (2.16) for the parts of Ψ which are even/odd under $(\lambda, C) \to (-\lambda, -C)$ are in agreement with the above identification of $\mu \pm Q$ as the respective Fermi levels.

Finally, we would like to make a minor comment concerning the nature of the flux appearing in (2.14) and (2.15). In general, type 0 string theories which are constructed with $\mathcal{N} = 2$ worldsheet supersymmetry contain twice as many RR fields as their type-II cousins. The middle-dimensional form has both electric and magnetic degrees of freedom. This doubling arises because the RR ground states of the NSR string with both even and odd worldsheet fermion number contribute. In contrast, the present two-dimensional type 0B theory actually only enjoys $\mathcal{N} = 1$ worldsheet supersymmetry, and the corresponding type-IIB model is dead (but see [23]). (One is tempted to call this the $\frac{0}{2}$B string.) It is no surprise, therefore, to find that there is only one kind of allowed RR background, which with our choice of variables is the electric background (2.4).

An analogous discussion holds for D-branes. Type 0B theories with $\mathcal{N} = 2$ worldsheet supersymmetry have two kinds of D-branes, conventionally called D$p+$ and D$p-$, each of which is charged (electrically) under the appropriate combination of RR fields [34, 35, 36]. The middle-dimensional form has one electrically and one magnetically charged brane. In our case, we have a reduction also of the number of branes because we only have $\mathcal{N} = 1$ worldsheet supersymmetry [6, 5].

To complete the story, we therefore have to justify that the RR form that couples to the tachyon on our unstable D0-branes is indeed the electric variable, and not possibly the magnetic one, $\tilde{C}$. Here, we can use the results of ref. [38], in which Sen's brane descent relations where studied for type 0 theories. According to [38] there are two completely independent descent charts, one for the + and one for the − branes. If we assume that the electric flux is the difference of Fermi levels, the instanton which corresponds to tunneling of eigenvalues is charged electrically under C. Therefore, since the instanton descends from the euclidean kink on the unstable D0-brane, we conclude that the unstable D0-brane must indeed couple to the electric variable C, as we had needed for consistency.

3. Computation of the potential

In our notation, we will mostly follow the usual conventions of [39, 40, 41], except that we will call Planck's constant $\hbar$ instead of $1/\beta$, reserving β for the inverse temperature appearing in section 4. In the double-scaling limit, we take $\hbar \to 0$, keeping various other quantities fixed. The Schrödinger equation for the matrix model eigenvalue λ is given by

$$\left(-\frac{\hbar^2}{2}\partial_\lambda^2 + V(\lambda) - \epsilon\right)\psi(\epsilon,\lambda) = 0 \tag{3.1}$$

where $V(\lambda)$ is the matrix model potential. In the double-scaling limit, only the quadratic behavior near the maximum is relevant. We will make the convenient choice [40]

$$V(\lambda) = \frac{1}{2}(1-\lambda^2)\,, \tag{3.2}$$

with infinite walls (Dirichlet boundary conditions on ψ) near $\lambda = \pm 1$. For comparison with [6], we note that we have chosen units in which $\alpha' = 1/2$.

It is convenient to introduce rescaled variables,

$$\lambda = \sqrt{\frac{\hbar}{2}}x \qquad \text{and} \qquad \epsilon = \frac{1}{2} - \hbar a\,, \tag{3.3}$$

in terms of which eq. (3.1) becomes

$$\left(\partial_x^2 + \frac{x^2}{4} - a\right)\psi(a,x) = 0\,. \tag{3.4}$$

This equation is solved by the parabolic cylinder functions called $W(a,x)$, $W(a,-x)$ in the conventions of [48].

The zero mode of C can easily be implemented by asymmetrically fine-tuning the positions of the walls in the limit $\hbar \to 0$. Explicitly, we impose Dirichlet boundary conditions $\psi(a,x_R) = \psi(a,x_L) = 0$, where

$$\begin{aligned} x_L &= -\sqrt{\frac{2}{\hbar}} + \sqrt{\frac{\hbar}{2}}C \\ x_R &= \sqrt{\frac{2}{\hbar}} + \sqrt{\frac{\hbar}{2}}C\,. \end{aligned} \tag{3.5}$$

(This corresponds to $\lambda_L = -1 + \hbar C/2$ and $\lambda_R = 1 + \hbar C/2$.) One can easily check that shifting the walls in this way shifts the perturbative energy levels as in (2.10). The eigenvalue equation we have to solve then becomes

$$W(a,x_L)W(a,-x_R) = W(a,-x_L)W(a,x_R)\,. \tag{3.6}$$

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By utilizing the asymptotic expansion of $W(a,x)$ and $W(a,-x)$ given in [48], we can reduce eq. (3.6) to the form

$$\frac{1}{k}\sin\varphi_L\sin\varphi_R = k\cos\varphi_L\cos\varphi_R\,, \tag{3.7}$$

up to terms of order $\hbar$. In (3.7),

$$\begin{aligned}\varphi_L &= \frac{1}{4}x_L^2 - a\ln(-x_L) + \frac{\pi}{4} + \frac{1}{2}\Phi_2\\ \varphi_R &= \frac{1}{4}x_R^2 - a\ln x_R + \frac{\pi}{4} + \frac{1}{2}\Phi_2\,,\end{aligned} \tag{3.8}$$

and

$$k = \sqrt{1+e^{2\pi a}} - e^{\pi a} \tag{3.9}$$

$$\Phi_2 = \arg\Gamma\left(\frac{1}{2}+ia\right)\,. \tag{3.10}$$

A trigonometric identity brings (3.7) into the form

$$\cos\bigl(\varphi_R-\varphi_L\bigr) = \sqrt{1+e^{-2\pi a}}\,\cos\bigl(\varphi_R+\varphi_L\bigr)\,, \tag{3.11}$$

which upon using the definitions (3.8) and (3.5) collapses to

$$\cos C = \sqrt{1+e^{-2\pi a}}\,\cos 2\varphi_0\,, \tag{3.12}$$

where

$$2\varphi_0 = \varphi_L+\varphi_R = \frac{1}{2}x_0^2 - 2a\ln x_0 + \frac{\pi}{2} + \Phi_2\,, \tag{3.13}$$

and $x_0 = \sqrt{2/\hbar}$, and we need only retain terms up to that order in $\hbar$. Eq. (3.12) determines the dependence of the eigenvalues a of (3.4) on C. Solutions of this equation come in pairs,

$$\varphi_0(a_\pm) = \mp\frac{1}{2}\arccos\left[\frac{\cos C}{\sqrt{1+e^{-2\pi a_\pm}}}\right] + n\pi\,, \tag{3.14}$$

for each integer n. By explicitly constructing the wavefunctions associated with these eigenvalues, one can check that the solutions a_+/a_- correspond to even/odd modes, respectively, where we remind the reader that we qualify modes as even or odd depending on their parity under the combined reflection $(x,C)\to(-x,-C)$. We also recall that we can use (3.14) to compute the density of states of the inverted harmonic oscillator. Summing even and odd modes, the density of pairs is given up to terms of order $1/|\ln\hbar|$ in the double-scaling limit by

$$\rho(a) = \left|\frac{dn}{da}\right| = \frac{1}{\pi}\varphi_0'(a)\,. \tag{3.15}$$

In the limit, ρ diverges as $|\ln\hbar|$.

To proceed, we parameterize the solutions of (3.14) as

$$a_\pm = a_0 + \frac{1}{2}\bigl(a_1 \pm a_2\bigr)\,, \tag{3.16}$$

where we define a_0 by the property $\varphi_0(a_0) = n\pi$, i.e., a_0 is independent of C, and a_1 and a_2 vanish in the double-scaling limit as $1/|\ln\hbar|^2$ and $1/|\ln\hbar|$, respectively, see eq. (3.19). We can then expand (3.14) for fixed a_0 to find

$$\varphi_0'(a_1 \pm a_2) = \mp v - \frac{1}{2} v' a_2\,, \tag{3.17}$$

where everything is a function of a_0 and

$$v = v(a) = \arccos\left[\frac{\cos C}{\sqrt{1+e^{-2\pi a}}}\right]. \tag{3.18}$$

Solving (3.17) yields the leading corrections to the eigenvalues in the double-scaling limit,

$$\begin{aligned} \varphi_0' a_2 &= -v(a_0) \\ \varphi_0' a_1 &= -\tfrac{1}{2} v' a_2\,. \end{aligned} \tag{3.19}$$

We have plotted this dependence of the eigenvalues (3.16) on C in figure 3 on page 8. The plot shows $-a(C)$ for the first few eigenvalues closest to the top of the potential.

We are now in a position to evaluate the finite contribution to the ground state energy of our matrix model that depends on the zero mode of the RR scalar C. As we have explained in the previous section, we turn on RR flux by filling even modes up to $\mu_+ = \mu + Q$ and odd modes up to $\mu_- = \mu - Q$ from the top of the potential, similarly to figure 2 on page 6. The ground state energy is then given by summing over all the eigenvalues (3.14) in the parameterization (3.16) (recall that in the conventions of this section, a is counted from the top of the potential downwards)

$$\sum_{\mu_+}^{\infty} a_+ + \sum_{\mu_-}^{\infty} a_- = \sum_{\mu}^{\infty}(2a_0 + a_1) + \left(\sum_{\mu_-}^{\mu} - \sum_{\mu}^{\mu_+}\right)\left(a_0 + \frac{1}{2}a_1\right) - \sum_{\mu_-}^{\mu_+}\frac{1}{2}a_2\,. \tag{3.20}$$

The rationale behind this particular splitting is the following. The first term in (3.20) is independent of the flux. It has the usual divergences computed in [39]. The second term depends on the flux, but is independent of C. It has a divergence which represents the bulk contribution of the background flux (2.4). We will analyze these contributions in more detail in the next section. The last term in (3.20) is the one of present interest. By plugging in the various definitions and using the density of states (3.15), we find that this term is finite in the double-scaling limit and given by

$$\begin{aligned} V(C) &= -\int_{\mu_-}^{\mu_+} da_0\ \rho(a_0)\ \frac{1}{2}a_2(a_0) = -\frac{1}{2\pi}\int_{\mu_-}^{\mu_+} da\ \varphi_0'(a) a_2(a) \\ &= \frac{1}{2\pi}\int_{\mu-Q}^{\mu+Q} da\ \arccos\left[\frac{\cos C}{\sqrt{1+e^{-2\pi a}}}\right], \end{aligned} \tag{3.21}$$

which is the main result of our paper. We show a plot of this function in figure 4.

The term (3.21) is the only C-dependence that survives the double-scaling limit. By curiosity, one may also ask for the next subleading term. We can easily extract this from

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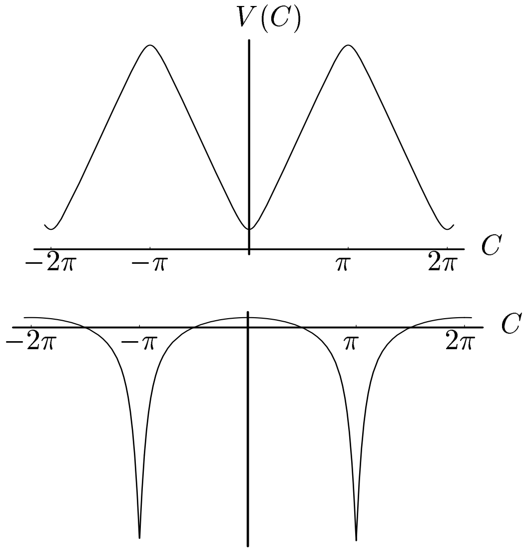

Figure 4: The ground state energy of the matrix model depends on the boundary condition asymmetry C. On the left, we plot the contribution (3.21) which is finite in the double-scaling limit, for $\mu = .4$ and $Q = .1$. On the right, we show the subleading term (3.22) which goes to zero as $1/|\ln\hbar|$.

the first term in (3.20) and find it to be

$$\frac{1}{4\pi|\ln\hbar|}\left(\arccos\left[\frac{\cos C}{\sqrt{1+e^{-2\pi a}}}\right]\right)^2\Bigg|_{\mu}^{\infty} . \tag{3.22}$$

In the perturbative limit $\mu \to \infty$, (3.21) obviously reduces to the result (2.11) found in the previous section. The only exceptions are the regions of order $e^{-\pi\mu}$ around the minimum and maximum of $V(C)$, where tunneling becomes dominant. The expression $V_{\rm pert}$ in (2.11) becomes invalid there, so the reader should not worry that (2.11) is odd under $C \to -C$, while (3.21) is even. As we have explained, the non-perturbative effects modify the identification of Q in the matrix model. It is curious to note that these non-perturbative effects actually lead to a large second derivative $V'' \sim e^{\pi\mu}$ at the minimum of the potential, in other words, the "mass" of the C-field appears to be of order e^{1/g_s}! We emphasize, however, that the quantity $V(C)$ we have computed cannot strictly be thought of as a mass term for C. In the spacetime interpretation, this potential energy is finite only after integrating over space, see eq. (2.12). Thus, the instanton contribution to the energy density vanishes, as could have been expected based on the fact that the instantons in question are D-branes localized at $\phi \to \infty$ in the Liouville picture [47, 15]. On the other hand, we note from the action (2.1) that the kinetic term for the C-field goes to zero at $\phi \to \infty$, so that even an infinitesimal potential can conceivably lead to a dynamical stabilization of the constant part of C. More precisely, if we assume that the potential indeed comes from the instantons localized at infinite ϕ, we can imagine cutting out a finite part of the Liouville direction at large ϕ where the kinetic term for C is very small. In this region, there would be a finite energy density from $V(C)$, which would stabilize the zero mode of C (its value at the cutoff) at the minimum of the potential.

Another interesting feature of $V(C)$ is that its minimum is at $C = 0$ or $C = \pi$ depending on the sign of Q. This seems to be a property of non-perturbative potentials for axion-like fields also in higher dimensions, as for example the ones considered in [2].

The term (3.22) is also interesting. First of all, its minimum is at $C = \pi$, which is somewhat unexpected. (For symmetry reasons, the minimum can only be at $C = 0$ or $C = \pi$.) Moreover, although it vanishes in the double-scaling limit, one can check that its second derivative with respect to C is in fact infinite at the minimum $C = \pi$. Thus, even though $V(C)$ itself vanishes at $Q = 0$, this subleading term seems to fix C with high precision at $C = \pi$ also in the absence of flux. This is true in particular if we imagine working with a cutoff Liouville direction.

We have here computed the exact answer for the potential from the matrix model point of view. It would naturally be extremely interesting to understand some of the features of the potential from the spacetime perspective.

Throughout the discussion, we have assumed that both Fermi levels are well below the top of the potential, i.e., $\mu > 0$, and $|Q| \ll \mu$). Only then is C a perturbative symmetry of the matrix model. When continuing μ to negative values (but keeping $|Q| \ll |\mu|$), Q should be interpreted as the magnetic flux (2.5), or alternatively as electric flux of the dual scalar $\tilde{C}$. Our formulas for $V(C)$ still make sense in that case, but the perturbative symmetry is the one associated with $\tilde{C}$. We compute the potential $V(\tilde{C})$ in appendix A.

4. Finite temperature partition function and dualities

We will here follow the usual route developed in [39] for extracting finite thermodynamic quantities from the double-scaled matrix model. In the matrix model (at finite N), we may consider the partition function

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N))}\,, \tag{4.1}$$

where $\beta = 1/T$ is the inverse temperature and μ is the chemical potential. We note that we are here using rescaled variables that stay fixed in the double-scaling limit. Thus, H in (4.1) is the Schrödinger operator (3.4), and $\mu > 0$ is measured from the top of the potential as in (3.3). We will parameterize $\beta = 2\pi R$, as is appropriate when thinking of (4.1) as the path-integral with euclidean time direction compactified on a circle of radius R.

An important quantity for defining the thermodynamic limit of interest for string theory is the mean particle number

$$\Delta = \langle N \rangle = \frac{1}{Z}\,\mathrm{Tr}\, N e^{-\beta(H-\mu N)} = \frac{\partial F}{\partial \mu}\,, \tag{4.2}$$

where $F = \ln Z/\beta$ is the grand-canonical free energy. More precisely, the double-scaling limit which defines non-critical string theory involves $\hbar \to 0$, $\Delta \to \infty$, keeping the chemical potential μ fixed. Since we are dealing with free fermions, it is easy to write Δ in the thermodynamic limit as

$$\Delta = \int_{-\infty}^{\infty} da\; \rho(a) f(a)\,, \tag{4.3}$$

where ρ is the density of states and f is the usual Fermi distribution function

$$f(a) = f(a,\mu) = \frac{1}{1+e^{2\pi R(\mu-a)}}\,. \tag{4.4}$$

The diverging expression (4.3) for Δ can be studied more conveniently by taking two derivatives with respect to μ. Using that $\partial f/\partial\mu = -\partial f/\partial a$ and $f'(a) \to 0$ for $a \to \pm\infty$, one finds

$$\frac{\partial^2\Delta}{\partial\mu^2} = \int da\ \rho'(a) f'(a)\,. \tag{4.5}$$

We briefly review the case of the bosonic string. Using (3.15) and the integral representation of the digamma function (F is often known as ψ, but we have already used that letter),

$$\rho'(a) = \frac{1}{2\pi}\,\mathrm{Im}\,F'\left(\frac{1}{2}+ia\right) = \frac{1}{2\pi}\int_0^\infty dt\,\frac{t/2}{\sinh t/2}\sin at\,, \tag{4.6}$$

as well as

$$f'(a) = \frac{\pi R}{2}\frac{1}{\cosh^2 \pi R(\mu - a)}\,, \tag{4.7}$$

we can write (4.5) as a "Fourier integral"

$$\frac{\partial^2\Delta}{\partial\mu^2} = \frac{1}{2\pi\mu}\int_0^\infty dt\,\frac{t/2R\mu}{\sinh t/2R\mu}\frac{t/2\mu}{\sinh t/2\mu}\sin t\,. \tag{4.8}$$

For comparison with [6], we note that we are here working with units in which $\alpha' = 1$, as follows from the fact that the curvature of the matrix model potential (3.2) is $V'' = -1$, which is identified with the mass of the tachyon $m^2 = -1/\alpha'$. The result (4.8) is invariant under T-duality,

$$R \to R' = \frac{\alpha'}{R}\,, \qquad \mu \to \mu' = \frac{R}{\sqrt{\alpha'}}\mu\,. \tag{4.9}$$

More precisely, recalling that Δ is related to the partition function via three derivatives with respect to μ and a factor of R, we see that

$$Z(R,\mu) = Z(R',\mu') \tag{4.10}$$

holds for the bosonic matrix model.

When studying the 0B theory, we may introduce two different chemical potentials μ_+ and μ_- for even and odd modes, respectively. As before, $\mu_\pm = \mu \pm Q$, where Q is related to the RR flux background as we have explained in section 2. For the computation of the partition function, one has to be slightly careful to split the various contributions in the right way. Starting at finite N, we have

$$\Delta = \sum\big(f(a_+,\mu_+) + f(a_-,\mu_-)\big)\,, \tag{4.11}$$

where the sum is over all eigenvalues which are given by the solutions of (3.12). One way to evaluate (4.11) is to work with different "densities of states" for even and odd modes, defined by differentiating (3.14). On the other hand, there is an even mode for every odd mode, so that the two densities should actually be the same. The answer is identical, but we find it cleaner to proceed by using the parameterization in (3.16). We obtain

$$\begin{aligned}\Delta &= \sum \big(f(a_0,\mu_+) + f(a_0,\mu_-)\big) + \sum \frac{1}{2} a_2 \big(f'(a_0,\mu_+) - f'(a_0,\mu_-)\big) \\ &= \int da\, \rho(a) \big(f(a,\mu_+) + f(a,\mu_-)\big) + \int da\, \frac{1}{2\pi} v(a) \big(f'(a,\mu_+) - f'(a,\mu_-)\big)\,, \end{aligned} \tag{4.12}$$

where we use the same density of states ρ as in the bosonic case (3.15). We can now again apply two derivatives to (4.12), and use the representation,

$$\frac{1}{2\pi} v''(a) = \frac{\pi}{4} \frac{\sinh \pi a}{\cosh^2 \pi a} = \frac{1}{2\pi} \int_0^\infty dt\, \frac{t/2}{\cosh t/2} \sin at\,. \tag{4.13}$$

In this expression, we have set C to the minimum of its potential ($0/\pi$ depending on the sign of Q). It is possible to compute the Fourier transform of $v'(a)$ also for general C, but we relegate the somewhat cumbersome expressions to the appendix. We now find

$$\begin{aligned}\frac{\partial^2 \Delta}{\partial \mu^2} &= \frac{1}{2\pi} \int_0^\infty \frac{t/2R}{\sinh t/2R} \left[\frac{t/2}{\sinh t/2} \mathrm{Im}\big(e^{i\mu_+ t} + e^{i\mu_- t}\big) + \frac{t/2}{\cosh t/2} \mathrm{Im}\big(e^{i\mu_+ t} - e^{i\mu_- t}\big) \right] \\ &= \frac{1}{\pi} \int_0^\infty dt\, \frac{t/2R}{\sinh t/2R} \left[\frac{t/2}{\sinh t/2} \sin \mu t\, \cos Qt + \frac{t/2}{\cosh t/2} \cos \mu t\, \sin Qt \right]. \end{aligned} \tag{4.14}$$

For $Q = 0$, and remembering that $\alpha' = 1/2$ in our conventions for the supersymmetric model, this reduces to the expression given in [6]. The reader might wonder about the second term in the square brackets in (4.14), which appears to be odd under $Q \to -Q$, whereas the partition function should naively not depend on the sign of the flux. The resolution of this puzzle is that we have not included the dependence on the zero mode of C, the minimum of whose potential switches from $C = 0$ to $C = \pi$ as we change the sign of Q. Taking this into account makes (4.14) symmetric under $Q \to -Q$. (See eq. (B.8) in the appendix.)

The result (4.14) is the complete non-perturbative result for the partition function as a function of μ, Q (and C). One can think of the first term in the square brackets as the non-perturbative resummation of the asymptotic series in $1/\mu^2$ that is defined in string perturbation theory. In fact, this asymptotic series is essentially the sum of the two series

$$\Delta''_{\text{pert}}(\mu, Q) = \Delta''_{\text{pert}}(\mu_+) + \Delta''_{\text{pert}}(\mu_-)\,, \tag{4.15}$$

where the perturbative expansion of $\Delta''_{\text{pert}}(\mu)$ is identical to the one of the bosonic string [39]

$$\begin{aligned}\Delta''_{\text{pert}}(\mu) &= \frac{1}{2\pi} \int_0^\infty dt\, \frac{t/2R}{\sinh t/2R} \frac{t/2}{\sinh t/2} \mathrm{Im}\, e^{i\mu t} \\ &\sim \frac{-1}{2\pi} \sum_{m=0}^\infty \frac{1}{\mu^{2m+1}(4R)^m} (2m)! \sum_{k=0}^m |2^{2k} - 2||2^{2m-2k} - 2| \frac{|B_{2k}||B_{2m-2k}|}{(2k)!(2m-2k)!} R^{m-2k}\,, \end{aligned} \tag{4.16}$$

where the B_{2k} are the Bernoulli numbers. On the other hand, one can see that the second term in square brackets in (4.14) is analytic in μ and its asymptotic expansion in $1/\mu^2$ vanishes, thus indicating its non-perturbative nature. To make this explicit, we write

$$\Delta''_{\text{np}}(\mu, Q) = \Delta''_{\text{np}}(\mu_+) - \Delta''_{\text{np}}(\mu_-) \tag{4.17}$$

with

$$\Delta''_{\rm np}(\mu) = \frac{1}{2\pi}\int_0^\infty dt\ , \frac{t/2R}{\sinh t/2R}\frac{t/2}{\cosh t/2}\,{\rm Im}\, e^{i\mu t}\,. \tag{4.18}$$

The essential feature of this integral is that the integrand is even under $t \to -t$. We can then close the integration contour in the upper/lower half plane (depending on the sign of μ) and write the integral as a sum over poles. Explicitly we have for $\mu > 0$,

$$\Delta''_{\rm np}(\mu) = \sum_{n=1}^{\infty}\left\{(-1)^{n+1}\frac{n^2\pi^2R^2}{\cos n\pi R}e^{-2\pi n\mu R} + (-1)^{n+1}\frac{(n-\frac{1}{2})^2\pi^2/R}{\sin(n-\frac{1}{2})\pi/R}e^{-2\pi(n-\frac{1}{2})\mu}\right\}\,, \tag{4.19}$$

and a similar expression for $\mu < 0$. We see from (4.19) that there are two types of non-perturbative contributions. One type comes from the poles of the $\frac{1}{\cosh t/2}$ factor in (4.18) and begins at order $e^{-\pi\mu}$.[3] The spacetime interpretation of these contributions are D-instantons, i.e., ZZ-brane [47] in the Liouville direction together with Dirichlet boundary condition in the euclidean time direction. The other type of contributions comes from the poles of the thermal factor in (4.18) and begins at order $e^{-2\pi R\mu}$. Obviously, the spacetime interpretation of this term is a D0-brane winding around the compact euclidean time direction, i.e., ZZ-brane together with Neumann boundary condition on the circle. It is interesting to note that something special happens to the structure of the non-perturbative series when the radius R is rational. In that situation, the two types of poles collide and combine into a single contribution (this limit is built into (4.19) automatically). It would be nice to understand this structure from the Liouville point of view. The relation between ZZ-branes and non-perturbative contributions in the *bosonic* matrix model has also been discussed in [19].

We stress that it is really sensible to write an instanton expansion for this second term in (4.14) because it is odd under $Q \to -Q$ and hence vanishes at $Q = 0$. The corresponding amplitudes vanish in perturbation theory, and receive contributions only from the instantons. This property is in distinction to the first term in (4.14), whose perturbative expansion is non-zero. From this asymptotic expansion, one can merely reconstruct that its non-perturbative ambiguities set in at order $e^{-2\pi\mu R}$ and $e^{-2\pi\mu}$, but it is not sensible to write an instanton expansion for these ambiguities.

With these results in hand, we now proceed to study various dualities that we expect our theory to enjoy. We begin with T-duality. The partition function of type 0A theory has been computed in [6] using the matrix model. It is found to be

$$\frac{\partial^2\Delta}{\partial\mu^2} = \frac{1}{2\pi}\int_0^\infty dt\, \frac{t}{\sinh t}\frac{t/2R}{\sinh t/2R}\sin\mu t\, e^{-|q|t}\,, \tag{4.20}$$

where q is the integer flux parameter of the 0A theory measuring the net number of D0-branes. Is is easy to see that the results for $q = Q = 0$ are related by the simple T-duality

$$Z_A(R_A, \mu_A, q = 0) = Z_B(R_B, \mu_B, Q = 0)\,, \tag{4.21}$$

[3]Interestingly, for $C \neq 0$, these contributions start at order $e^{-2\pi\mu}$, see appendix B.

where the parameters are related by

$$R_A = \frac{1}{2R_B} = \frac{\alpha'}{R_B}\,, \qquad \mu_A = 2R_B\mu_B = \frac{R_B}{\sqrt{\alpha'/2}}\,\mu_B\,. \tag{4.22}$$

It is worthwhile to point out that (4.22) is a slightly non-standard transformation rule on the string coupling, which in the usual type-II context transforms as in (4.9). It would be interesting to understand why the string coupling has to be normalized in a different way in this two-dimensional type 0 context.

If we want to check T-duality also for non-zero flux, the form of the above expressions suggests that we should try to Wick rotate $q \to iQ$ and supplement (4.22) with

$$iq = Q_A = Q_B\frac{\mu_A}{\mu_B} = \frac{\sqrt{2\alpha'}}{R_A}\,. \tag{4.23}$$

The fact that the flux transforms with a factor of i is quite suggestive of a timelike T-duality. Namely, recall that the flux-parameter Q in type 0B is for an electric flux, which is a timelike one-form. Rotating to euclidean time, the flux becomes imaginary. T-duality turns the one-form into a "zero-form flux", which is really just the dual of the 2-form flux of 0A. But now undoing the analytical continuation does not give back the factor of i, so that the flux stays imaginary. We will then take (4.22) and (4.23) as a hypothesis for T-duality, and compute the partition function for the 0A model at imaginary $q = iQ$. We will be able to match the perturbative expansions of the two theories, but we will not be able to find agreement at the non-perturbative level.

In the type 0A matrix model introduced in [6], the Schrödinger equation for the eigenvalues (the analogue of (3.4)) is (see also [43, 44, 45] for early studies of this matrix model potential)

$$\left(\frac{1}{x}\partial_x x\partial_x + \frac{x^2}{4} - \frac{q^2}{x^2} - a\right)\psi(a,x) = 0\,. \tag{4.24}$$

Writing $z = \frac{ix^2}{2}$ and $\psi = z^{q/2}e^{-z/2}f$, we obtain the confluent hypergeometric equation

$$\left(z\partial_z^2 + (1+q-z)\partial_z - \left(\frac{1}{2}+\frac{q}{2}-\frac{ia}{2}\right)\right)f = 0\,, \tag{4.25}$$

with well-known solutions. The problem with the analytical continuation $q \to iQ$ comes from the boundary condition at $x = 0$. Indeed, the two solutions of (4.24) behave as $\psi \sim x^\alpha$ with $\alpha^2 = q^2$ near $x = 0$. We see that for $q^2 \geq 1$, only one of the solutions is normalizable, $\int_0 x^{2\alpha+1} < \infty$ and there is no need for a boundary condition. For $q^2 < 1$, the singularity is of limit-circle type and we need an extra boundary condition to make sure that the differential operator (4.24) is self-adjoint. The usual procedure for $q^2 = 0$ is to require that the Laplace operator in the plane be self-adjoint. But for $q^2 = -Q^2 < 0$, the solutions are actually oscillatory near $x = 0$. The possible real boundary conditions come in a one-parameter family,

$$\psi \sim e^{i\delta/2}x^{iQ} + e^{-i\delta/2}x^{-iQ} \qquad \text{as } x \sim 0\text{, with } \delta \text{ real.} \tag{4.26}$$

Using the traditional Kummer notation for the confluent hypergeometric function, we can write the solution of (4.24) satisfying the boundary conditions (4.26) as

$$\begin{aligned}\psi(a,x) &= e^{i\delta/2} x^{iQ} e^{-z/2} \Phi\left(\frac{1}{2}+\frac{iQ}{2}-\frac{ia}{2}, 1+iQ; z\right)+\\ &\quad + e^{-i\delta/2} x^{-iQ} e^{-z/2} \Phi\left(\frac{1}{2}-\frac{iQ}{2}-\frac{ia}{2}, 1-iQ; z\right). \end{aligned} \tag{4.27}$$

By working through the asymptotics of the Φ function, one finds that this solution behaves near $x \to \infty$ as

$$\psi(a,x) \sim \text{const.}\ \frac{1}{x}\cos\left(\frac{x^2}{4} - a\ln\frac{x}{\sqrt{2}} - \frac{\pi}{4} - \varphi\right), \tag{4.28}$$

where the phase shift φ is given by

$$\varphi = \arg\left[e^{i(\delta+Q\ln 2)/2}\frac{\Gamma(1+iQ)}{\Gamma\left(\frac{1}{2}+\frac{iQ}{2}+\frac{ia}{2}\right)}e^{-\pi Q/4} + e^{-i(\delta+Q\ln 2)/2}\frac{\Gamma(1-iQ)}{\Gamma\left(\frac{1}{2}-\frac{iQ}{2}+\frac{ia}{2}\right)}e^{\pi Q/4}\right] \tag{4.29}$$

After absorbing an a-independent phase into δ, this reduces to

$$\varphi = \arg\left[\frac{e^{i\delta/2}e^{-\pi Q/4}}{\Gamma\left(\frac{1}{2}+\frac{iQ}{2}+\frac{ia}{2}\right)} + \frac{e^{-i\delta/2}e^{\pi Q/4}}{\Gamma\left(\frac{1}{2}-\frac{iQ}{2}+\frac{ia}{2}\right)}\right]. \tag{4.30}$$

It seems hard to find an explicit integral representation for the density of states $\frac{1}{\pi}\frac{d\varphi}{da}$ similar to (4.6). But noticing that

$$\left|\frac{\Gamma\left(\frac{1}{2}-\frac{iQ}{2}+\frac{ia}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{iQ}{2}+\frac{ia}{2}\right)}e^{i\delta}e^{-\pi Q/2}\right|^2 = \frac{e^{\pi a}+e^{-\pi Q}}{e^{\pi a}+e^{\pi Q}}, \tag{4.31}$$

we can expand the problem for large $\mu \gg 1$. We write $\varphi = \varphi_1 + \varphi_2$ as the sum of two terms

$$\begin{aligned}\varphi_1 &= -\frac{1}{2}\left[\arg\Gamma\left(\frac{1}{2}+\frac{iQ}{2}+\frac{ia}{2}\right)+\arg\Gamma\left(\frac{1}{2}-\frac{iQ}{2}+\frac{ia}{2}\right)\right]\\ \varphi_2 &= \arg[w+w^{-1}], \end{aligned} \tag{4.32}$$

where $w = \left[e^{i\delta}e^{-\pi Q/2}\Gamma\left(\frac{1}{2}-\frac{iQ}{2}+\frac{ia}{2}\right)/\Gamma\left(\frac{1}{2}+\frac{iQ}{2}+\frac{ia}{2}\right)\right]^{1/2}$ has absolute value exponentially close to one. The first term in (4.32) gives after the usual steps

$$\frac{\partial^2\Delta}{\partial\mu^2} \approx \frac{1}{2\pi}\int_0^\infty dt\,\frac{t}{\sinh t}\frac{t/2R}{\sinh t/2R}\sin\mu t\cos Qt\,, \tag{4.33}$$

which is precisely the T-dual of the first term in (4.14), as we have claimed. This fact that the asymptotic expansions of the two partition functions agree to all order in perturbation theory can also be seen by analytically continuing directly the result (4.20). Indeed, as we have seen above, only the even part of e^{-qt} will lead to a perturbative series. Thus, the perturbative expansion can be obtained by replacing the exponential with $\cosh qt$, which leads to a convergent integral for sufficiently small q. Now rotating $q \to iQ$ directly shows the agreement with the first term in (4.14). But of course, it is safer to first analytically continue the problem as we have done above.

What about the non-perturbative contributions? On the 0A side, these come from the second term in (4.32). We find for large $a \gg Q$

$$\varphi_2 \sim e^{-\pi a} \sinh \pi Q \, \tan\left(\frac{\delta}{2} - \frac{Q}{2} \ln a\right), \tag{4.34}$$

where we have again absorbed all a-independent phases into δ. We see that the first non-perturbative contribution to the 0A partition function at imaginary flux $Q = iq$ is of order $\sim e^{-\pi\mu} \sinh \pi Q$ and depends in a somewhat strange way on the boundary condition δ. This result can indeed be matched with the order of the first correction in (4.19) after using the T-duality transformation rules (4.22). The fact that we need the additional boundary condition at $x = 0$ is suggestive of trying to match it with C. We have computed the corresponding dependence of the first instanton contribution on C from the formulas in the appendix but have not been able to match this with the dependence of (4.34) on δ. And the prefactors of the exponentials do not show an exact match either.

Moreover, one has to ask for the presence of the second type of non-perturbative contributions, which begin at order $e^{-\pi\mu}$ on the 0B side. On the 0A side, this would be a contribution of order $e^{-\pi\mu R}$ and has to come from the thermal density factor $1/(1+e^{2\pi R(\mu-a)})$. But these contributions start at order $e^{-2\pi R\mu}$, in disagreement with expectations. It therefore seems unlikely to us that the naive prescriptions that we have followed above can reveal an exact T-duality between 0A and 0B, once the flux is turned on and non-perturbative effects are accounted for.

Non-perturbative violations of T-duality have been noticed before, as for example in [49]. In this paper, the duality group of the heterotic string on K3$\times T^2$ is studied. It is argued that the failure of naive T-duality at the non-perturbative level should be viewed not as a breakdown of perturbative intuition, but rather as a deformation of the latter. In the present context, this would suggest to deform the T-duality relations (4.22) by terms of order $e^{-\pi\mu}$ and $e^{-\pi\mu R}$ such that the partition functions agree exactly. It would be worthwhile to study this proposal further.

Before leaving the subject, we also wish to clarify what we mean by "the type 0A matrix model at imaginary q". The definition of the 0A matrix model in [6] uses rectangular $(N+q) \times N$ dimensional complex matrices, and a $\mathrm{U}(N+q) \times \mathrm{U}(N)$ gauge group which removes the phase of the "eigenvalues". The holographic intuition behind this definition is that the matrix model is the worldvolume theory of a large number of $D0$ and $\overline{D0}$ branes, and q is the net $D0$-brane charge. Leaving aside the fact that this theory is rather unphysical to study at imaginary q, it is not even clear how to define the matrix model mathematically! What we have in mind here is an alternative definition of the 0A matrix model, in which the q leftover $D0$-branes have been dissolved into background flux. Similarly to the discussion at the end of section 2, the tachyon $\mathcal{T}$ on a general $D_p\overline{D}_p$ system, which is a complex field, couples to the RR $C^{(p-1)}$ form potential such that a vortex of $\mathcal{T}$ carries one unit of lower-dimensional charge. This corresponds to a term $\oint d \arg \mathcal{T} \wedge C^{(p-1)} = \int d(d \arg \mathcal{T}) \wedge C^{(p-1)}$ in the worldvolume action. Integrating by parts, this becomes $\int d \arg \mathcal{T} \wedge F^{(p)}$. Thus, in the type 0A theory in two dimensions, if we turn on two-form flux $F^{(2)} = *F^{(0)} \propto q$, we obtain a term $q \int \frac{d}{dt}(\arg \lambda)$ in the quantum mechanics of the complex eigenvalues λ. It is easy to see that after gauge fixing, such a term corresponds precisely to a shift of the angular momentum in the eigenvalue plane by q units, just as in (4.24). This way of defining the 0A model does not pose any obvious obstacles to $q \to iQ$, except for the fact that it might be difficult to define the model at finite N because the hamiltonian is unbounded from below.

Another interesting duality [6] of the type 0B theory in two dimensions is "S-duality" $\mu \to -\mu$ and $C \to \tilde{C}$, as we have mentioned several times already. Our results for the partition function at $C = 0$ are obviously invariant under $\mu \to -\mu$, $Q \to -Q$ which is simply particle-hole duality of the matrix model eigenvalues. It is somewhat less clear what becomes of our non-perturbative results involving the zero mode of C. While the formulas do not suffer any obvious pathology, C as described in section 2 is not a perturbative symmetry for fermions above the top of the potential. For such states, the perturbative energy levels are left/right moving, and non-perturbative effects are reflections off the potential. The perturbative symmetry associated with these energy levels is precisely the zero mode of the dual variable $\tilde{C}$, and one can check that the non-perturbative dependence of eigenvalues on $\tilde{C}$ is precisely as in (3.14), with $a \to -a$. This is discussed in detail in appendix A.

We can also notice yet another duality from our matrix model computations. The partition function (which is the third integral of (4.14) with respect to μ) is also invariant under $\mu_- \to -\mu_-$, i.e., applying particle-hole duality to the odd modes only. In terms of μ and Q, this corresponds to exchanging NS and RR backgrounds $\mu \leftrightarrow Q$, a rather intriguing duality indeed! The T-dual of this statement was noticed in in [8], where the duality of [46] was reinterpreted in this fashion in the 0A context. The duality is much more manifest in the 0B model.

To summarize, the 0B matrix model depends on two parameters μ and Q, or equivalently μ_+ and μ_-, but models which differ only by a sign of $\mu_\pm$ are equivalent to each other. If $|Q| \ll |\mu|$, the matrix model is the non-perturbative definition of a 0B non-critical string theory. Depending on the sign of μ, Q is interpreted as either electric or magnetic flux with corresponding perturbative gauge symmetry C or $\tilde{C}$. It would be interesting to understand whether there is also a perturbative string theory description in the case $|\mu| \ll |Q|$. An analog of the perturbative gauge symmetry does not seem to exist in that case.

5. Bosonization and amplitudes in RR flux background

According to our identification in section 2, the matrix model with different Fermi levels for even and odd modes provides a dual description of type-0B string theory in the RR flux background (2.4). This formulation is extremely simple and makes it possible to compute scattering amplitudes and a complete S-matrix for this string theory. In this section, we will describe the formalism for studying these amplitudes, leaving a detailed study of their properties for future work. Our approach follows the older works [40, 41] on the bosonic string and the more recent paper [7] on the type 0B model (with vanishing RR flux). The use of matrix models to study non-critical strings in flux backgrounds was first proposed in the context of type 0A in [8].

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In relating matrix quantum mechanics to spacetime physics in two dimensions, one identifies collective excitations of the Fermi sea of matrix eigenvalues with the bosonic fields of the corresponding string theory. This identification includes a non-local field redefinition which is implemented by the so-called "leg-pole factors" in the scattering amplitudes, but we can ignore this subtlety for our purposes. For the 0B model, it was proposed in [5, 6] to identify the even fluctuations of the Fermi surface with the tachyon field T and odd fluctuations with the RR scalar C. In [7], it was pointed out that unitarity of the S-matrix requires the inclusion of "solitonic sectors" into the Hilbert space of asymptotic states. These solitonic sectors are created by two additional bosonic fields, which we will call S^+ and S^-, which carry C-field charge, and have no perturbative spacetime interpretation.

More precisely, as in [40], one begins by introducing second quantized fermion fields

$$\Psi(t,\lambda) = \int_{-\infty}^{\infty} e^{i\omega t} b_\epsilon(\omega)\psi^\epsilon(\omega,\lambda)\,, \tag{5.1}$$

where the $\psi^\epsilon(\omega,\lambda)$ are a complete basis of solutions of the classical field equation (3.1) or (3.4), and where we identify $\omega = -a$ to conform with other conventions in the literature. For fixed ω, different such bases are appropriate in different situations. Of primary interest for defining scattering amplitudes are in/out bases which are defined by requiring specific asymptotic behavior. We will denote solutions which are in/outgoing on the left and right of the potential by subscripts L and R, respectively. The 'in' and 'out' bases are related, as usual, by the S-matrix,

$$\begin{pmatrix} b_{\mathrm{L}}^{\mathrm{out}} \\ b_{\mathrm{R}}^{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} b_{\mathrm{L}}^{\mathrm{in}} \\ b_{\mathrm{R}}^{\mathrm{in}} \end{pmatrix}, \tag{5.2}$$

where we have chosen the phases of L and R modes such that the $\mathbb{Z}_2$ symmetry of the potential is manifest (i.e., S commutes with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). By textbook methods, one finds the reflection and transmission coefficients to be

$$R(\omega) = \frac{i}{\sqrt{2\pi}} e^{-\pi\omega/2} \Gamma\left(\frac{1}{2} - i\omega\right), \qquad T(\omega) = i e^{\pi\omega} R(\omega)\,. \tag{5.3}$$

We stress that (5.1) is the full "interacting field", where in the present case the only interactions are with the external potential, and does not depend on the conventional choice of basis. For example, for the purpose of defining the vacuum, it will be mandatory to introduce other bases of modes with fixed parity.

But let us for the moment focus on the asymptotic left/right bases. Both in and out modes yield the usual algebra of creation and annihilation operators,

$$\begin{aligned} \{b_{\mathrm{L}}(\omega), b_{\mathrm{L}}^\dagger(\omega')\} &= \delta(\omega-\omega') \\ \{b_{\mathrm{R}}(\omega), b_{\mathrm{R}}^\dagger(\omega')\} &= \delta(\omega-\omega') \\ \text{and} \qquad \{b_\epsilon^\#(\omega), b_\epsilon^\#(\omega')\} &= 0 \qquad \text{for all other combinations.} \end{aligned} \tag{5.4}$$

To complete the definition of the theory, we have to fix a representation of this algebra, as we will do a little later. But independent of this representation, we can introduce bosonic operators as fermion bilinears such as

$$
\begin{aligned}
a_{\mathrm{LL}}(\omega) &= \int_{-\infty}^{\infty} b_{\mathrm{L}}^{\dagger}(\xi) b_{\mathrm{L}}(\xi+\omega) \\
a_{\mathrm{RR}}(\omega) &= \int_{-\infty}^{\infty} b_{\mathrm{R}}^{\dagger}(\xi) b_{\mathrm{R}}(\xi+\omega) \,.
\end{aligned} \tag{5.5}
$$

As usual, these bosonic operators satisfy the relations of a current algebra[4]

$$
\begin{aligned}
[a_{\mathrm{LL}}(\omega_1), a_{\mathrm{LL}}(\omega_2)] &= \omega_1 \delta(\omega_1+\omega_2) \\
[a_{\mathrm{RR}}(\omega_1), a_{\mathrm{RR}}(\omega_2)] &= \omega_1 \delta(\omega_1+\omega_2) \,.
\end{aligned} \tag{5.6}
$$

In [5, 6], it was proposed to identify the spacetime tachyon field T with the even combination $a_{\mathrm{LL}} + a_{\mathrm{RR}}$ and the RR scalar with the odd combination $a_{\mathrm{RR}} - a_{\mathrm{LL}}$. But as pointed out in [7], the bosonic S-matrix computed with these fields cannot possibly be unitary. Indeed, it is easy to see that we can define two further fermion bilinears,

$$
\begin{aligned}
a_{\mathrm{LR}}(\omega) &= \int_{-\infty}^{\infty} b_{\mathrm{L}}^{\dagger}(\xi) b_{\mathrm{R}}(\xi+\omega) \\
a_{\mathrm{RL}}(\omega) &= \int_{-\infty}^{\infty} b_{\mathrm{R}}^{\dagger}(\xi) b_{\mathrm{L}}(\xi+\omega) \,,
\end{aligned} \tag{5.7}
$$

which complete (5.6) to a $\mathrm{U}(2)$ current algebra

$$
\begin{aligned}
[a_{\mathrm{RR}}(\omega_1), a_{\mathrm{RL}}(\omega_2)] &= a_{\mathrm{RL}}(\omega_1+\omega_2) \\
[a_{\mathrm{LL}}(\omega_1), a_{\mathrm{RL}}(\omega_2)] &= -a_{\mathrm{RL}}(\omega_1+\omega_2) \\
[a_{\mathrm{RL}}(\omega_1), a_{\mathrm{RL}}(\omega_2)] &= 0 \\
[a_{\mathrm{RL}}(\omega_1), a_{\mathrm{LR}}(\omega_2)] &= a_{\mathrm{RR}}(\omega_1+\omega_2) - a_{\mathrm{LL}}(\omega_1+\omega_2) + \omega_1\delta(\omega_1+\omega_2) \,, \qquad \text{etc.}
\end{aligned} \tag{5.8}
$$

In this algebra, the tachyon $T \sim a_{\mathrm{LL}} + a_{\mathrm{RR}}$ is one of the $\mathrm{U}(1)$'s and the RR scalar $C \sim a_{\mathrm{RR}} - a_{\mathrm{LL}}$ together with the "soliton operators" $S^{+} \sim a_{\mathrm{LR}}$ and $S^{-} \sim a_{\mathrm{RL}}$ generate an $\mathrm{SU}(2)$ algebra. The main point of [7] is that because the interactions (scattering of fermions off the potential) do not respect this symmetry algebra, it is inconsistent to restrict the Hilbert space of asymptotic states to a fixed charge under the $\mathrm{U}(1)^2$ currents (5.6).

Going back to the fermionic picture, we recall that we still have to specify the vacuum of our theory. This amounts to splitting the fermionic algebra (5.4) into creation and annihilation operators, in other words, the choice of a Fermi level. As we have explained in detail in section 2, the only non-perturbatively stable possibility is to choose separately Fermi levels for the even and odd modes, related to the L/R modes by

$$
b_{\pm}(\omega) = \frac{1}{\sqrt{2}}\big(b_{\mathrm{L}}(\omega) \pm b_{\mathrm{R}}(\omega)\big) \,. \tag{5.9}
$$

[4]Actually, obtaining this algebra requires appropriately regulating in the UV, as is the case in our situation.

Thus, we choose $\mu_+ = \mu + Q$, $\mu_- = \mu - Q$, and assume for simplicity that $\mu_+ < 0$ and $\mu_- < 0$. The vacuum $|0\rangle$ is defined by[5]

$$\begin{aligned} b_\pm(\omega)|0\rangle &= 0 \qquad \omega > \mu_\pm \,, \\ b_\pm^\dagger(\omega)|0\rangle &= 0 \qquad \omega < \mu_\pm \,. \end{aligned} \tag{5.10}$$

As we have mentioned, for $\mu_+ = \mu_-$, the bosons defined in (5.5) have the interpretation of collective excitations of the Fermi surfaces which are coming in from the left or from the right, while the off-diagonal bosons (5.7) create solitonic sectors. For $Q \neq 0$, we loose a good semiclassical picture of the ground state, so this interpretation is not so obvious anymore.

Finally, we write out explicitly the two-point functions computed using the definitions (5.5), (5.7) with respect to the vacuum defined in (5.10). The formulae are simplest in terms of $a_{\pm\pm} = b_\pm^\dagger b_\pm$, which can be related to T, C, S^+ and S^- by working through the definitions (5.9). We find

$$\begin{aligned} \langle a_{++}^{\text{out}}(\omega_1) a_{++}^{\text{in}}(\omega_2)\rangle &= \delta(\omega_1+\omega_2)\int_{\mu_+-\omega_1}^{\mu_+} d\xi_1\; R_+(\xi_1+\omega_1)R_+^*(\xi_1) \\ \langle a_{+-}^{\text{out}}(\omega_1) a_{-+}^{\text{in}}(\omega_2)\rangle &= \delta(\omega_1+\omega_2)\int_{\mu_--\omega_1}^{\mu_+} d\xi_1\; R_-(\xi_1+\omega_1)R_+^*(\xi_1) \\ \langle a_{+-}^{\text{out}}(\omega_1) a_{+-}^{\text{in}}(\omega_2)\rangle &= 0\,, \qquad \text{etc.}\,, \end{aligned} \tag{5.11}$$

where the $R_\pm$ are the eigenvalues of the free fermion S-matrix (5.2)

$$R_\pm = R \pm T = \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2} - i\omega\right)\left(ie^{-\pi\omega/2} \mp e^{\pi\omega/2}\right) \tag{5.12}$$

6. Open problems

In this paper, we have studied aspects of the RR sector of type 0B string theory in two dimensions, using the matrix model as a non-perturbative definition. The crucial step was the precise identification, following [6], of the RR flux background as the difference of Fermi levels for even and odd modes of the inverted harmonic oscillator potential of the double-scaled matrix model. We have also identified the zero mode of the RR scalar C in the finite-N matrix model as the mismatch of the perturbative (in $\hbar$) energy levels on the two sides of the potential.

We have then given three applications of this identification of RR fields in the matrix model. Firstly, we have computed the potential energy for the zero mode of C, which is induced by turning on flux, and stabilized by non-perturbative effects. Our result is the complete answer to this question, the analogue of which in higher dimensions is of great interest for the construction of realistic string theory vacua. Secondly, we have computed the finite-temperature partition function of the 0B string in the flux background. We have seen that T-duality with the 0A model is realized at the perturbative level, but

[5] In the conventions of this section, $\mu < 0$ when the Fermi levels are below the top of the potential.

appears to be violated by non-perturbative effects. Thirdly, we have outlined how one can use the matrix model to study scattering amplitudes of 0B strings in RR backgrounds, following [40, 41, 8, 7]. Understanding strings in RR backgrounds is also an important problem in higher-dimensional string theories.

To conclude this paper, we list here a few open questions which have been raised by our results.

- Quite obviously, it would be extremely interesting to understand the spacetime or string theory computation of our result for the potential $V(C)$. There, the origin of the potential are non-perturbative effects associated with D-instantons, and it would be interesting to see how they contribute to $V(C)$.
- More generally, it would be interesting to compute some other terms in the effective action and to see the structure of non-perturbative effects, which could in particular provide guidance for similar questions in higher dimensions.
- It would be nice to understand better what happens to T-duality with 0A for non-vanishing flux.
- The matrix model gives complete control over interpolations between different dual perturbative string theories. It would be worthwhile to understand what happens at the transition region, e.g., when μ_- goes through zero at finite string coupling.
- What is the analytical behavior of scattering amplitudes in non-zero RR background? Can one derive the identification of RR fields directly from a discretization of super-Riemann surfaces?
- According to [50], the old-fashioned $c = 1$ string at the self-dual radius is related to the topological string on the conifold. It would be interesting to see whether our model with unequal chemical potentials corresponding to non-zero RR background could have a similar interpretation in the context of topological strings.

Needless to say, we hope to come back to some of these problems in the future.

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A. $V(\tilde{C}) = V(C)$

This discussion is *mutatis mutandis* almost identical to the one given in section 2 and 3. When the chemical potential μ is above the top of the matrix model potential, the perturbative energy levels are left/right-moving moving waves instead of localized on one side of the potential. The magnetic gauge symmetry associated with $\tilde{C}$ shifts the phase of left movers by $e^{i\tilde{C}}$ and the phase of right movers by $e^{-i\tilde{C}}$. We can implement $\tilde{C}$ in the finite-N matrix model by imposing periodic boundary conditions on the wavefunctions with a phase shift of $e^{i\tilde{C}}$. This means that we study the Schrödinger equation (3.4) and impose

$$\begin{aligned} e^{i\tilde{C}/2}\psi(a,x_0) &= e^{-i\tilde{C}/2}\psi(a,-x_0) \\ e^{i\tilde{C}/2}\psi'(a,x_0) &= e^{-i\tilde{C}/2}\psi'(a,-x_0)\,, \end{aligned} \tag{A.1}$$

where $x_0 = \sqrt{2/\hbar} \to \infty$ in the double-scaling limit. An alternative is to study the equation

$$\left[\left(\partial_x - i\frac{\tilde{C}}{2x_0}\right)^2 + \frac{x^2}{4} - a\right]\psi(a,x) = 0\,. \tag{A.2}$$

with strictly periodic boundary conditions on ψ. This description is of course gauge equivalent to (A.1), but makes it more manifest that left/right movers pick up a phase shift when traversing the potential from $-x_0$ to x_0. This implementation of $\tilde{C}$ is appropriate if we construct the matrix model using unitary matrices with periodic potential.

Upon writing the general solution of (3.4) in terms of parabolic cylinder functions $W(a,x)$ and $W(a,-x)$, eq. (A.1) becomes the eigenvalue equation

$$\begin{vmatrix} e^{i\tilde{C}/2}W(a,x) - e^{-i\tilde{C}/2}W(a,-x) & e^{i\tilde{C}/2}W(a,-x) - e^{-i\tilde{C}/2}W(a,x) \\ e^{i\tilde{C}/2}\partial_x W(a,x) + e^{-i\tilde{C}/2}\partial_x\big(W(a,-x)\big) & e^{i\tilde{C}/2}\partial_x\big(W(a,-x)\big) + e^{-i\tilde{C}/2}\partial_x W(a,x) \end{vmatrix} = 0\,. \tag{A.3}$$

Using the asymptotics for $x \gg |a|$,

$$W(a,x) \sim \sqrt{\frac{2k}{x}}\cos\varphi \qquad W(a,-x) \sim \sqrt{\frac{2}{kx}}\sin\varphi \tag{A.4}$$

with $\varphi = \frac{1}{4}x^2 - a\ln x + \frac{\pi}{4} + \frac{1}{2}\Phi_2$ and $k = \sqrt{1+e^{2\pi a}} - e^{\pi a}$, we obtain from (A.3)

$$\cos\tilde{C} = \sqrt{1+e^{2\pi a}}\sin 2\varphi_0\,, \tag{A.5}$$

where $\varphi_0 = \varphi$ for $x = x_0 = \sqrt{2/\hbar}$. Up to a shift of φ_0 by $\pi/4$, this is exactly the "S-dual" of (3.12) and shows that the non-perturbative potential one would compute for the magnetic variable $\tilde{C}$ is identical to the one we found for the electric variable C. Which potential is sensible to compute depends on the sign of μ, i.e., whether we interpret Q a electric or magnetic flux from the point of view of perturbative string theory.

B. $V(C)$ at finite temperature

In this appendix, we compute the Fourier transform of $v'(a)$ in (3.18) in order to obtain a closed form expression for the non-perturbative piece of the 0B partition function at finite temperature and for general C. The quantity of interest is

$$\Delta'_{\rm np}(\mu) = \int_{-\infty}^{\infty} da\, \frac{1}{2\pi}v'(a)f'(a,\mu)\,, \tag{B.1}$$

where

$$f'(a,\mu)=\frac{\pi R}{2}\frac{1}{\cosh^2\pi R(\mu-a)}=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\,e^{i(\mu-a)t}\frac{t/2R}{\sinh t/2R}\,, \tag{B.2}$$

and

$$\frac{1}{2\pi}v'(a)=\frac{e^{-2\pi a}}{1+e^{-2\pi a}}\frac{\cos C}{\sqrt{\sin^2 C+e^{-2\pi a}}}\,. \tag{B.3}$$

The integrand in

$$\int_{-\infty}^{\infty}da\;e^{iat}\frac{e^{-2\pi a}}{1+e^{-2\pi a}}\frac{\cos C}{\sqrt{\sin^2 C+e^{-2\pi a}}} \tag{B.4}$$

has poles at $e^{-2\pi a}=-1$ and branch points at $e^{-2\pi a}=-\sin^2 C$. It is convenient to place the cuts for $n\geq 0$ from $a=\left(n+\frac{1}{2}\right)i-\frac{1}{2\pi}\ln\sin^2 C$ to $a=\left(n+\frac{1}{2}\right)i+\infty$. We can then evaluate the integral (B.4) for $t>0$ by closing the contour in the upper half plane, picking up the poles on the imaginary axis and integrating back and forth along the cuts. Paying attention to signs, we find the contribution from the poles to be

$$\begin{aligned}\sum_{n=0}^{\infty}2\pi i\frac{-1}{2\pi}e^{-(n+\frac{1}{2})t}\frac{\cos C}{\sqrt{\sin^2 C-1}}&=\frac{\cos C}{|\cos C|}\sum_{n=0}^{\infty}(-1)^n e^{-(n+\frac{1}{2})t}\\&=\frac{\cos C}{|\cos C|}\frac{1/2}{\cosh t/2}\,.\end{aligned} \tag{B.5}$$

To evaluate the contribution from the cuts, we substitute $a=\left(n+\frac{1}{2}\right)i-\frac{1}{2\pi}\ln\sin^2 C-\frac{1}{2\pi}\ln z$, with $z\in[0,1]$, and are reduced to

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_0^1\frac{dz}{z}\frac{-2}{2\pi}(-1)^n e^{-(n+\frac{1}{2})t}\left(\sin^2 C\right)^{-\frac{it}{2\pi}}z^{-\frac{it}{2\pi}}\frac{-z\sin^2 C}{1-z\sin^2 C}\frac{\cos C}{|\sin C|\sqrt{1-z}}=\\&\quad=\frac{1}{\pi}\frac{t/2}{\cosh t/2}|\sin C|\cos C\left(\sin^2 C\right)^{-\frac{it}{2\pi}}\int_0^1 dz\,z^{-\frac{it}{2\pi}}(1-z)^{-1/2}(1-z\sin^2 C)^{-1}\,.\end{aligned} \tag{B.6}$$

We recognize the remaining integral as the particular hypergeometric function

$$\begin{aligned}&\frac{\Gamma\left(1-\frac{it}{2\pi}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{it}{2\pi}\right)}F\left(1,1-\frac{it}{2\pi};\frac{3}{2}-\frac{it}{2\pi};\sin^2 C\right)=\\&\qquad=\left(\frac{1}{2}-\frac{it}{2\pi}\right)\frac{(\sin^2 C)^{\frac{it}{2\pi}}}{|\cos C|\,|\sin C|}B\left(1-\frac{it}{2\pi},\frac{1}{2}\right)B_{\sin^2 C}\left(\frac{1}{2}-\frac{it}{2\pi},\frac{1}{2}\right)\end{aligned} \tag{B.7}$$

which defines the incomplete Beta function according to the usual sources. Pulling things together, we can finally write

$$\Delta'_{\rm np}(\mu)=\frac{1}{2\pi}\,\mathrm{Re}\int_0^{\infty}dt\,e^{-i\mu t}\frac{t/2R}{\sinh t/2R}\frac{1/2}{\cosh t/2}\frac{\cos C}{|\cos C|}\left[1+g(t,C)\right], \tag{B.8}$$

where

$$g(t,C)=\frac{1}{\pi}\left(\frac{1}{2}-\frac{it}{2\pi}\right)B\left(1-\frac{it}{2\pi},\frac{1}{2}\right)B_{\sin^2 C}\left(\frac{1}{2}-\frac{it}{2\pi},\frac{1}{2}\right). \tag{B.9}$$

The structure of the non-perturbative instanton contributions that follows from these calculations is quite interesting. As we have seen in the main text (4.19), the first instanton term for $C = 0$ is of order $e^{-\pi\mu}$. On the other hand, when C is non-zero, it can be seen from (B.4) that non-perturbative effects set in at order $e^{-2\pi\mu}$ for $\mu \to \infty$. (But diverge as $1/\sin C$ for $C \to 0$.) This behavior is easy to understand in the matrix model. When $C = 0$, the perturbative energy levels on the left and right match, and the simple tunneling amplitude is of order $e^{-\pi\mu}$. When $C \neq 0$, the perturbative levels do not match, and one has to tunnel forth and back to find a non-zero contribution.

C. $\hat{c} = 1$ at $R = 1$

In the bosonic $c = 1$ matrix model, many formulas simplify at the self-dual radius $R = 1$. The supersymmetric model is not mapped onto itself under $R \to 1/R$, as we have seen in the main text. But we can again evaluate some formulas more explicitly when we set $R = 1$. For example, the perturbative part of the finite-temperature partition function is essentially the bosonic string result.

$$\begin{aligned}\Delta''_{\rm pert}(\mu) &= \frac{1}{2\pi}\int_0^\infty dt\, \left(\frac{t/2}{\sinh t/2}\right)^2 \sin\mu t \\ &= \frac{1}{2\pi}\,\mathrm{Re}\;\partial_\mu^2\big(\mu\,\digamma(1-i\mu)\big)\,,\end{aligned} \tag{C.1}$$

leading to the well-known asymptotic expansion

$$2\pi\Delta''_{\rm pert}(\mu) \sim \frac{1}{\mu} + \sum_{n=1}^{\infty}\frac{(2n-1)|B_{2n}|}{\mu^{2n+1}}\,. \tag{C.2}$$

In the present case, we of course have to add together two such terms with $\mu = \mu_\pm$.

We can also compute the non-perturbative part of the partition function exactly at $R = 1$. Consider (again for one of $\mu_\pm$)

$$\begin{aligned}\Delta_{\rm np}(\mu) &= \int_{-\infty}^{\infty} da\, \frac{1}{2\pi} v'(a) f(a,\mu) \\ &= \int_{-\infty}^{\infty} da\, \frac{1}{1+e^{2\pi(\mu-a)}}\,\frac{e^{-2\pi a}}{1+e^{-2\pi a}}\,\frac{\cos C}{\sqrt{\sin^2 C + e^{-2\pi a}}}\,,\end{aligned}$$

which setting $y = e^{2\pi\mu}$ and $z = e^{-2\pi a}$ can be reduced to an elementary integral

$$\begin{aligned}&= \frac{1}{2\pi}\int_0^\infty dz\, \frac{1}{1+yz}\,\frac{1}{1+z}\,\frac{\cos C}{\sqrt{\sin^2 C + z}} \\ &= \frac{1}{2\pi}\frac{\cos C}{|\cos C|}\frac{1}{y-1}\left[2\arcsin|\sin C| - \pi - \sqrt{y}|\cos C|\frac{2\arcsin\big(|\sin C|\sqrt{y}\big)-\pi}{\sqrt{1-y\sin^2 C}}\right],\end{aligned} \tag{C.3}$$

where an appropriate branch of the arcsin is understood once $y\sin^2 C > 1$.

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